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Due date: June 8, 2026 at 8:00 p.m.

Algorithmic Game Theory

Summer Term 2026

Exercise Set 5

If you would like to submit your solutions for this problem set, please send them via email to aheuser1@uni-bonn.de by Monday evening. Submitting solutions in groups is also possible.

If you would like to present one of your solutions in class, please use the following link to book a presentation slot by Monday evening:
https://terminplaner6.dfn.de/b/7370115f4b6e1c679209e2f6a01fb948-1756661
A short meeting to discuss your solution is mandatory before presenting it in class. To book a time slot for this meeting, please use the following link by Monday evening as well:
https://terminplaner6.dfn.de/b/127c3e54b7f4877a3a1b0c9b08699381-1756670

Exercise 1:
We call s an ϵ-approximation to a pure Nash equilibrium if $c_i(s) \leq (1+\epsilon)c_i(s'_i, s_{-i})$ for all i and s'_i.

(a) Consider a (λ, μ)-smooth cost-minimization game and let and $0 < \epsilon < \frac{1}{\mu} - 1$. Prove that the PoA of ϵ-approximations to pure Nash equilibria is at most $\frac{(1+\epsilon)\lambda}{1-(1+\epsilon)\mu}$.

(b) Can you state a similar result for more general equilibrium concepts?

Exercise 2:
Consider a second-price auction with a fixed value profile $(v_i)_{i \in N}$. Since the value profile is fixed, we get a normal-form utility-maximization game.

(a) Show that there exists a pure Nash equilibrium in the defined game.

(b) Now, consider a game in which only two players participate and $v_1 \gg v_2$ holds. Prove that even in this setting there exists a pure Nash equilibrium such that bidder 2 wins.

Exercise 3:
We consider an auction of k identical items. Each bidder can acquire at most one of the items. If bidder i gets one of the items, she has a value of v_i. Otherwise, that is, if she does not get an item, she has a value of 0.

(a) State a generalization of the second-price auction and prove that it is truthful (the second-price auction covers the case of $k = 1$). Follow steps in the spirit of Lecture 10.

(b) Now, consider a mechanism which sequentially performs k second-price auctions. That is, initially each bidder reports one bid. Then, in each auction, one item is sold among the remaining players using their initial bids. Show that truthful bidding does not necessarily lead to a pure Nash equilibrium even in the special case of three players and $k = 2$.

Exercise 4:
A billionaire considers selling tours to the moon. The cost of building a rocket is C. Let $N = \{1, \ldots, n\}$ be the set of people who initially have declared an interest in the trip. The billionaire wishes to design a mechanism that will recover his cost but does not have information about the private valuation the bidders have for joining the trip. Therefore, he runs the following auction given as pseudocode:

- All bidders $i \in N$ simultaneously submit their bids $b_i \geq 0$.
- $S \leftarrow N$
- While $S \neq \emptyset$ do
  - $S' \leftarrow \{i \in S \mid b_i \geq \frac{C}{|S|}\}$
  - If $S' = S$, then allocate a seat for each $i \in S$ and no seat for each $i \in N \setminus S$. All bidders $i \in S$ have to pay $\frac{C}{|S|}$. The rest of the bidders $i \in N \setminus S$ has to pay nothing. Return.
  - Otherwise, $S \leftarrow S'$
- Do not allocate any seat and charge no payments at all. Return.

Show that the described mechanism is truthful.

(a) Show that the described mechanism is truthful.

(b) Show that if the bidders are truthful, the auction finds the largest set of bidders that can share the target cost C equally, if there is one.

Exercise 5:
Recall the *Greedy-by-Value* and *Greedy-by-Sqrt-Value-Density* algorithms for single-minded combinatorial auctions of lecture 12. Let us analyse another greedy algorithm that looks as follows.

Greedy-by-Value-Density

- Re-order the bids such that $\frac{b_1^*}{|S_1^*|} \geq \frac{b_2^*}{|S_2^*|} \geq \cdots \geq \frac{b_n^*}{|S_n^*|}$.
- Initialize the set of winning bidders to $W = \emptyset$.
- For $i = 1$ to n do: If $S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset$, then $W = W \cup \{i\}$.

Let $d = \max_{i \in \mathcal{N}} |S_i^*|$. Show that the given algorithm yields a d-approximation.