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Due date: In the lecture on 22nd May 2019, 12:15

Algorithmic Game Theory and the Internet

Summer Term 2019

Exercise Set 7

Exercise 1: (2 Points)
Recall the auction of k identical items from Exercise Set 6: each bidder can acquire at most one of the items. If bidder i gets one of the items, she has a value of v_i. Otherwise, that is, if she does not get an item, she has a value of 0.

Make use of Myerson's Lemma in order to design a mechanism that is truthful. For this purpose, explicitly state the function f, verify that it is monotone, and calculate the payment rule p resulting from the integral formula.

Exercise 2: (3+1 Points)
A billionaire is considering selling tours to the moon. The cost of building a rocket is C. Let $N_0 = \{1, \ldots, n\}$ be the set of people who initially have declared an interest in the trip. The billionaire wishes to design a mechanism that will recover his cost but does not have information about the private valuation the bidders have for joining the trip. Therefore, he runs the following auction given as pseudocode:

- All bidders $i \in N_0$ simultaneously submit their bids $b_i \geq 0$.
- $N \leftarrow N_0$
- While $N \neq \emptyset$ do
  - $N' \leftarrow \{i \in N \mid b_i \geq \frac{C}{|N|}\}$
  - If $N' = N$, then allocate a seat for each $i \in N$ and no seat for each $i \in N_0 \setminus N$. All bidders $i \in N$ have to pay $\frac{C}{|N|}$. The rest of the bidders $i \in N_0 \setminus N$ has to pay nothing. Return.
  - Otherwise, $N \leftarrow N'$
- Do not allocate any seat and charge no payments at all. Return.

(a) Show that the described mechanism is truthful.

(b) Show that if the bidders are truthful, the auction finds the largest set of bidders that can share the target cost C equally, if there is one.

Exercise 3 and 4 on the next page.

Exercise 3: (5 Points)

Consider a *Knapsack Auction* which is defined the following way. Each bidder i has a publicly known weight w_i and a private value v_i. A feasible outcome is any set S of bidders such that $\sum_{i \in S} w_i \leq W$ holds for a fixed bound W. Furthermore, we assume that $0 \leq w_i \leq W$ for all bidder i.

The following algorithm yields a 2-approximation:

- Sort and renumber the bidders such that $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \ldots \frac{b_n}{w_n}$. Let k be the largest integer such that $\sum_{i=1}^{k} w_i \leq W$ and set $S_1 = \{1, \ldots, k\}$.
- Let i^* be the bidder with the maximum bid b_i among all bidders and set $S_2 = \{i^*\}$.
- Return the better solution of S_1 and S_2.

Show that the given algorithm is monotone and state a truthful mechanism with the aid of Myerson's Lemma.

Exercise 4: (3+3+3 Points)

In this exercise we want to prove an alternative characterization of truthful mechanisms which is stated in the following theorem. Moreover, this restated characterization holds for arbitrary mechanisms and not only for single-parameter mechanisms.

Theorem. A mechanism $M = (f, p)$ is truthful if and only if the following two conditions are met.

(i) For every pair b_i, b_i': If $f(b_i, b_{-i}) = f(b_i', b_{-i})$, then we also have $p_i(b_i, b_{-i}) = p_i(b_i', b_{-i})$. In other words: For all b_{-i}, for all $a \in X$ there exist prices $p_a \in \mathbb{R}$ such that for all b_i with $f(b_i, b_{-i}) = a$ we have $p_i(b_i, b_{-i}) = p_a$.

(ii) The mechanism optimizes for each player. Formally: For every pair b_i, b_{-i} the following holds
$$f(b_i, b_{-i}) \in \arg\max_{a \in A} (b_i(a) - p_a),$$
where the set of allocations A is equal to the image of $f(\cdot, b_{-i})$.

For this purpose, prove the following claims:

(a) If condition (i) is violated, then the mechanism M cannot be truthful.

(b) If condition (ii) is violated, then the mechanism M cannot be truthful.

(c) If conditions (i) and (ii) are met, then M is a truthful mechanism.

Hint: You should not use Myerson's Lemma in this task (since it is not helpful). Instead, it is sufficient to make use of the truthfulness inequality with reasonably chosen deviations. See also how we derived *payment difference sandwich* in the proof of Theorem 11.2.