Walrasian Equilibria

Thomas Kesselheim   Last Update: June 24, 2019

We have seen many different approaches to mechanism design so far, all talking about auctions in some form. The most common form of a mechanism, however, is very different: Whenever we go shopping, we are not asked to bid for items. Instead, each of them has a price tag. We may either buy the item at this price or leave it. Therefore, we now turn to the question how well such prices can coordinate markets. Today, we will start with some classic economic theory about this.

1 Setting and Definition

We consider the standard setting of combinatorial auctions. There are n bidders N and m items M. Feasible allocations are vectors $S = (S_1, \ldots, S_n)$, $S_i \subseteq M$ for all $i \in N$, and $S_i \cap S_{i'} = \emptyset$ for $i \neq i'$. Each bidder has a valuation function $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$. We consider *full information*. That is, the valuation functions are fixed and known.

A *Walrasian equilibrium* is an equilibrium in the sense that is a stable state. In contrast to the equilibrium concepts that we got to know so far, it does not talk about players' strategies but rather about prices making an allocation stable.

Definition 20.1. *A pair of a price vector $q \in \mathbb{R}_{\geq 0}^m$ and an allocation $S = (S_1, \ldots, S_n)$ is a Walrasian Equilibrium if*

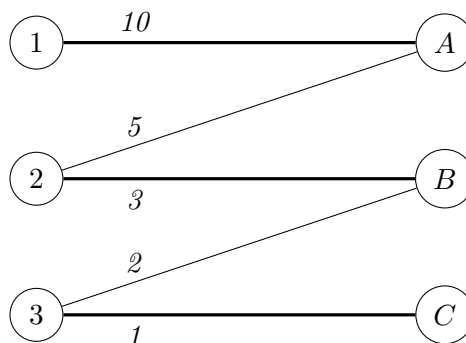(a) *Each bidder i gets a bundle that maximizes utility:*

$$v_i(S_i) - \sum_{j \in S_i} q_j \geq v_i(S_i') - \sum_{j \in S_i'} q_j \quad \text{for all } S_i' \subseteq M \ .$$

(b) *If an item j is unallocated, i.e., $j \notin \bigcup_{i \in N} S_i$, then $q_j = 0$.*

Example 20.2. *If there is only a single item and $v_1 \geq v_2 \geq \ldots \geq v_n$, then the Walrasian equilibria are exactly the prices $q_1 \in [v_1, v_2]$ paired with the allocation that assigns the item to bidder 1.*

Example 20.3. *We now consider multiple items with unit-demand valuations. That is, a bidder's valuation is of the form $v_i(S) = \max_{j \in S} v_{i,j}$ for $v_{i,j} \geq 0$. Assigning the items is just the same as finding a matching in a complete bipartite graphs whose vertices are $N \cup M$. The edge between $i \in N$ and $j \in M$ has weight $v_{i,j}$.*

We consider an example with three bidders 1, 2, 3 and three items A, B, C. We only draw edges of positive value.

The allocation is given by the thick edges. One choice for q would be $q_A = 3$, $q_B = 1$, $q_C = 0$. These are the prices that come out of the VCG payments, a connection that we will see later. But it is not the only feasible choice for q. An alternative would be $q_A = 10$, $q_B = 3$, $q_C = 1$.

Example 20.4. *There are profiles of valuation functions for which no Walrasian equilibrium exists. Consider the example of three single-minded bidders. Bidder 1 wants items 1 and 2, bidder 2 wants items 1 and 3, bidder 3 wants items 2 and 3. Each of them has a value of 1 for getting both items and 0 otherwise.*

We now have to distinguish multiple cases how the allocation $S = (S_1, S_2, S_3)$ is chosen. For no such allocation there is a price vector q that fulfills both conditions. Consider, for example, $S_1 = \{1, 2\}$, $S_2 = S_3 = \emptyset$. Then $q_3 = 0$ because it is not allocated. This means that $q_1 \geq 1$ because otherwise bidder 2 would not be happy. Analogously, $q_2 \geq 1$. This, however, means that $v_1(S_1) - \sum_{j \in S_1} q_j \leq 1 - 2 = -1 < 0 = v_1(\emptyset) - \sum_{j \in \emptyset} q_j$. This is a contradiction to condition (a).

2　First Welfare Theorem

Our first theorem is a very famous one: It tells us that the allocation of any Walrasian equilibrium maximizes social welfare. This has often been interpreted as "markets are efficient". Undoubtedly, this is a little questionable. One of many reasons is that Walrasian equilibria do not always exist.

Theorem 20.5. *If (q, S) is a Walrasian equilibrium, then S maximizes social welfare.*

Proof. Let $S^* = (S_1^*, \ldots, S_n^*)$ be an allocation that maximizes social welfare. Then for each bidder i we have

$$v_i(S_i) - \sum_{j \in S_i} q_j \geq v_i(S_i^*) - \sum_{j \in S_i^*} q_j \ .$$

Summing this inequality over all bidder i yields

$$\sum_{i \in N} v_i(S_i) - \sum_{i \in N} \sum_{j \in S_i} q_j \geq \sum_{i \in N} v_i(S_i^*) - \sum_{i \in N} \sum_{j \in S_i^*} q_j \ .$$

Observe that $\sum_{i \in N} \sum_{j \in S_i} q_j = \sum_{j \in M} q_j$ because each item is allocated at most once in S and items that are not allocated in S have a zero price by property (b). Furthermore $\sum_{i \in N} \sum_{j \in S_i^*} q_j \leq \sum_{j \in M} q_j$ because also in S^* each item is allocated at most once. Unallocated items may have a non-zero price but it cannot be negative. This directly implies

$$\sum_{i \in N} v_i(S_i) \geq \sum_{i \in N} v_i(S_i^*) \ ,$$

which means that S also maximizes social welfare.　　　　　　　　　　　　　　　　　　$\square$

If you are familiar with linear programming and duality, this argument might look familiar. Indeed, it is nothing but weak LP duality: The price vector q is a feasible solution to the dual LP that certifies optimality of S.

3　Unit-Demand VCG Outcome as Walrasian Equilibrium

As our second main result, we will now see an interesting connection between Walrasian equilibria and the VCG mechanism if bidders have unit-demand valuations. To simplify notation, we assume that there are more items than bidders. So, every bidder gets exactly one item.

Let S denote a social-welfare maximizing allocation. Let S^{-i} denote the same if bidder i is excluded. Recall that on truthful bids the VCG mechanism defines the payment of bidder i as $p_i(v) = \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i' \neq i} v_{i'}(S_{i'})$.

We use this to define item prices. If item j is unallocated in S, set $q_j = 0$. If item j is assigned to bidder i, set its price to bidder i's VCG payment. That is,

$$q_j = p_i(v) = \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i' \neq i} v_{i'}(S_{i'})$$

Theorem 20.6. *The price vector q defined by the VCG mechanism for unit-demand valuations combined with any social-welfare maximizing allocation is a Walrasian equilibrium.*

So, this means that in particular a Walrasian equilibrium always exists if valuations are unit-demand. To prove the theorem, we need an important lemma, which is also interesting on its own right. We let S^{+j} be a social-welfare optimizing allocation if there are *two copies* of j. To avoid issues of tie-breaking, let it be different from S only when the welfare is strictly higher than in S.

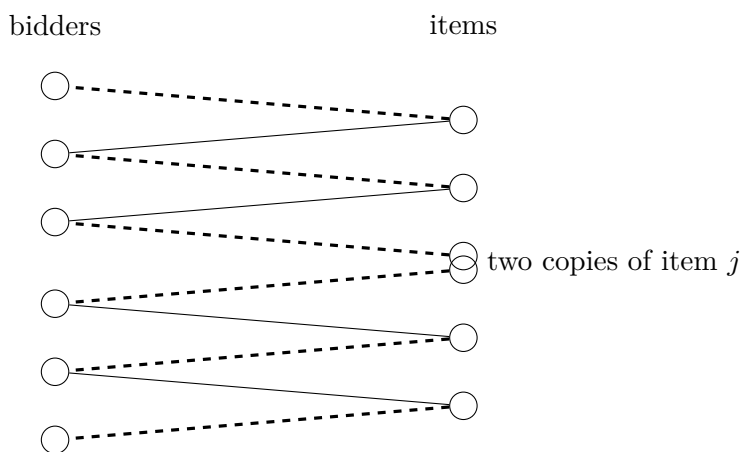**Lemma 20.7.** *For every item $j \in M$*

$$q_j = \sum_{i' \in N} v_{i'}(S_{i'}^{+j}) - \sum_{i' \in N} v_{i'}(S_{i'}) \ .$$

That is, instead of removing the bidder who gets the item, we might as well add another copy of it. This mirrors our intuition of VCG payments. A bidder has to pay by how much he hurts the others, that is, by taking away the item.

Proof. The proof relies on two intuitive facts regarding the allocation S^{+j}:

(i) If j is not allocated in S, neither copy is allocated in S^{+j}.

(ii) If j is allocated to bidder i in S, bidder i also receives one copy of j in S^{+j}.

Both actually should not be too surprising. One can prove them using common arguments about bipartite matching. Let us see a proof sketch for (i). We first observe that if S^{+j} allocates only one copy of j we could simply use it as S. This then contradicts the optimality of S or the definition of S^{+j}. If both copies are allocated, we can draw the following picture.

The solid edges indicate the assignment in S whereas the dashed one indicate the assignment in S^{+j}. Observe that we have two *disjoint* alternating paths starting at the two copies of j. They represent how S^{+j} gets its higher welfare compared to S. So, the overall weight of dashed edges is higher than the overall weight of solid edges. However, as we have two disjoint paths, this property has to hold for at least one of them as well. This means, we could improve S by allocating j once according to that path.

Coming to the proof of our lemma, we distinguish whether the item is allocated in S or not. If it is not allocated in S, then $q_j = 0$. Also, S and S^{+j} are identical by property (i).

If item j is allocated to bidder i in S, then $q_j = \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i' \neq i} v_{i'}(S_{i'})$. Allocation S^{-i} allocates each item at most once, not allocating anything to bidder i. By property (ii), the allocation S^{+j} has the same property if we leave out bidder i. Both maximize social welfare under this constraint, so their social welfare (not counting bidder i on either side) is the same: $\sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) = \sum_{i' \neq i} v_{i'}(S_{i'}^{+j})$.

This implies

$$
\begin{aligned}
q_j &= \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i' \neq i} v_{i'}(S_{i'}) = \sum_{i' \neq i} v_{i'}(S_{i'}^{+j}) - \sum_{i' \neq i} v_{i'}(S_{i'}) \\
&= \left(\sum_{i' \neq i} v_{i'}(S_{i'}^{+j}) + v_{i,j} \right) - \left(\sum_{i' \neq i} v_{i'}(S_{i'}) + v_{i,j} \right) \\
&= \sum_{i'} v_{i'}(S_{i'}^{+j}) - \sum_{i'} v_{i'}(S_{i'}) \ .
\end{aligned}
$$

So, also in this case the lemma holds. □

Proof of Theorem 20.6. By Lemma 20.7, we have

$$
q_\ell = \sum_{i'} v_{i'}(S_{i'}^{+\ell}) - \sum_{i'} v_{i'}(S_{i'}) \ .
$$

What is $\sum_{i'} v_{i'}(S_{i'}^{+\ell})$? It is the highest welfare that we can achieve if there is an extra copy of ℓ. One possible allocation in this case is to assign the new copy of item ℓ to bidder i and to make the other allocation according to S^{-i}. It may not be the best allocation but it is certainly a feasible one. Therefore

$$
\sum_{i'} v_{i'}(S_{i'}^{+\ell}) \geq v_{i,\ell} + \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) \ .
$$

This gives us a lower bound on q_ℓ

$$
q_\ell = \sum_{i'} v_{i'}(S_{i'}^{+\ell}) - \sum_{i'} v_{i'}(S_{i'}) \geq v_{i,\ell} + \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i'} v_{i'}(S_{i'}) \ .
$$

Note that

$$
\sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i'} v_{i'}(S_{i'}) = \sum_{i' \neq i} v_{i'}(S_{i'}^{-i}) - \sum_{i' \neq i} v_{i'}(S_{i'}) - v_{i,j} = q_j - v_{i,j} \ .
$$

So, in combination $q_\ell \geq v_{i,\ell} + q_j - v_{i,j}$, or equivalently $v_{i,j} - q_j \geq v_{i,\ell} - q_\ell$. □

4 VCG gives Smallest Walrasian Prices

Finally, we will show one more observation, namely that the Walrasian prices are always at least as high as the ones defined by VCG. That is VCG does not only give us a Walrasian equilibrium, it is also the smallest one.

Theorem 20.8. *Let (q, S) be any Walrasian equilibrium, q^{VCG} be the prices defined in the previous section. Then $q_j \geq q_j^{VCG}$ for all $j \in M$.*

Proof. Fix any $j \in M$. By property (a) of q, bidder i would not prefer S_i^{+j} to S_i under prices q. That is

$$
v_i(S_i) - \sum_{\ell \in S_i} q_\ell \geq v(S_i^{+j}) - \sum_{\ell \in S_i^{+j}} q_\ell \ .
$$

Taking the sum over all i, this yields

$$\sum_{i \in N} v_i(S_i) - \sum_{i \in N} \sum_{\ell \in S_i} q_\ell \geq \sum_{i \in N} v_i(S_i^{+j}) - \sum_{i \in N} \sum_{\ell \in S_i^{+j}} q_\ell \ .$$

Note that $\sum_{i \in N} \sum_{\ell \in S_i} q_\ell = \sum_{\ell \in M} q_\ell$ because each item occurs at most once and the ones that do not occur have zero price by property (b). Furthermore $\sum_{i \in N} \sum_{\ell \in S_i^{+j}} q_\ell \leq q_j + \sum_{\ell \in M} q_\ell$ because each item occurs at most once except of item j, which may occur a second time. This gives us

$$\sum_{i \in N} v_i(S_i) - \sum_{\ell \in M} q_\ell \geq \sum_{i \in N} v_i(S_i^{+j}) - \left(q_j + \sum_{\ell \in M} q_\ell \right) \ .$$

This is equivalent to

$$q_j \geq \sum_{i \in N} v_i(S_i^{+j}) - \sum_{i \in N} v_i(S_i) = q_j^{\mathrm{VCG}} \ . \qquad \square$$

Recommended Literature

- Tim Roughgarden's lecture notes `http://theory.stanford.edu/~tim/w14/l/l22.pdf` and lecture video `https://youtu.be/-xX1z5K5KkM`