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Due date: In the lecture on 25th June 2018, 10:15

# Algorithmic Game Theory and the Internet

Summer Term 2018

Exercise Set 10

**Exercise 1:** (4+4 Points)

Recall the valuation functions of single-minded bidders from Definition 13.2. Let the maximum bundle size be defined by $d = \max_{i \in \mathcal{N}} |S_i^*|$.

(a) Show that in the case of single-minded bidders with maximum bundle size $d$, item bidding with first price payments is $(\frac{1}{2}, 2d)$-smooth.

**Hint:** In order to define deviation bids $b_{i,j}^*$, consider a welfare-maximization allocation on $v$. If bidder $i$ does not get his bundle in the optimal allocation, then define $b_{i,j}^* = 0$ for all items $j \in M$. Otherwise, define $b_{i,j}^* = \frac{v_i}{2d}$ for all $j \in S_i^*$ and $b_{i,j}^* = 0$ if $j \notin S_i^*$. That is, each winner in the optimal allocation equally divides the value for his bundle among all items of the bundle and bids half of it.

(b) Now, we define prices for items as in Lecture 20 by setting

$$p_j^v = \begin{cases} \frac{1}{2d} v_i(S_i^*) & \text{if buyer } i \text{ gets item } j \text{ in optimal solution on } v \\ 0 & \text{if item } j \text{ is unassigned in optimal solution on } v \end{cases}$$

Show that using these prices in the full-information setting gives a $\frac{1}{2d}$-approximation of the optimal social welfare. (Like in Step 1 of Lecture 20)

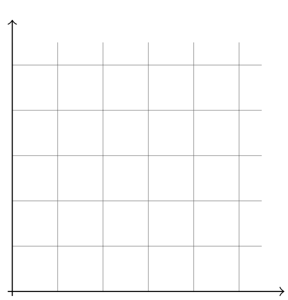
**Exercise 2:** (2+3 Points)

Consider three unit-demand buyers and two items with

$$v_{1,1} = 5, v_{1,2} = 3, v_{2,1} = 3, v_{2,2} = 4, v_{3,1} = 2, v_{3,2} = 2 \ .$$

(a) Determine the Walrasian price vector that as determined by the VCG mechanism.

(b) Now find *all* Walrasian price vectors $q$. (We know that the solution to (a) is componentwise smaller than any other such vector.) Draw these vectors in a coordinate system with axes $q_1$ and $q_2$.



Exercises 3 and 4 on the next page.

**Exercise 3:** (4 Points)

Consider $m$ items and $n$ unit-demand bidders. We define a generalization of Walrasian equilibria: Let $S$ be a matching of items to bidders and $q \in \mathbb{R}^m_{\geq 0}$ be a price vector. We call the pair $(q, S)$ an *$\epsilon$-approximate Walrasian equilibrium* if unallocated items have price 0, every bidder $i$ has non-negative utility $v_{i,S(i)} - q_{S(i)} \geq 0$, and every bidder receives an item within $\epsilon$ of its favorite, i.e., $v_{i,S(i)} - q_{S(i)} \geq v_{i,j} - q_j - \epsilon$ for every item $j$.

Prove an approximate version of the First Welfare Theorem: If $(q, S)$ is an $\epsilon$-approximate Walrasian equilibrium, then the social welfare of an optimal matching $S^*$ cannot surpass the one of $S$ by more than $\min\{m, n\} \cdot \epsilon$.

**Exercise 4:** (3 Points)

Have a look at the single-minded combinatorial auction with three bidders and items $a, b, c$ which is depicted below. State all values of $x \in \mathbb{R}_{\geq 0}$ such that there exists a Walrasian equilibrium and prove your claim.